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A note on the gauge equivalence between the Manin–Radul and Laberge–Mathieu super KdV hierarchies

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Abstract. The gauge equivalence between the Manin–Radul and Laberge–Mathieu super KdV hierarchies is revisited. Apart from the Inami–Kanno transformation, we show that there is another gauge transformation which also possesses the canonical property. We explore the relationship of these two gauge transformations from the Kupershmidt–Wilson theorem viewpoint and, as a by-product, obtain the Darboux–Bäcklund transformation for the Manin–Radul super KdV hierarchy. The geometrical intepretation of these transformations is also briefly discussed.

1. Introduction

Recently, Morosi and Pizzocchero [1–3] discussed the gauge equivalence of the Manin–Radul (MR) [4] and Laberge–Mathieu (LM) [5] super Korteweg–de Vries (sKdV) hierarchies from a bi-Hamiltonian and Lie superalgebraic viewpoint. This approach can be viewed as a superextension of the Drinfeld–Sokolov method [6] for building a KdV-type hierarchy for a simple Lie algebra. They showed [1] that the gauge transformation proposed by Inami and Kanno (IK) [7] not only preserves the Lax equations but also the bi-Hamiltonian structures corresponding to the MR and LM hierarchies. In particular, they provided a geometrical meaning of the IK transformation which rests on the natural fibred structure appearing in the bi-Hamiltonian reduction of loop superalgebras.

In this paper, in addition to the IK transformation, we find that there is another gauge transformation between the MR and LM sKdV hierarchies preserving the Lax equations. We investigate the canonical property of this gauge transformation and discuss the connection to the IK transformation from the Kupershmidt–Wilson theorem [8] viewpoint. As a by-product, the Darboux–Bäcklund transformation (DBT) for the MR sKdV hierarchy can be constructed from these two gauge transformations. The geometrical interpretation of these two transformations is also briefly discussed.

Our paper is organized as follows. In section 2 the bi-Hamiltonian structures of the MR and LM sKdV hierarchies are briefly reviewed. In section 3 we introduce a gauge transformation between these two hierarchies and investigate its canonical property. Then in section 4 we discuss the relationship between this transformation and the IK transformation from the KW theorem viewpoint. Concluding remarks are presented in section 5.

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2. Bi-Hamiltonian structures of the MR and LM sKdV hierarchies

The MR sKdV hierarchy was defined originally from the reduction of the MR super Kadomtsev–Petviashvili hierarchy [4]. It has the following Lax equation:

$$\partial_n L^{MR} = [B_n^{MR}, L^{MR}] \tag{2.1}$$

with

$$L^{MR} = \partial^2 - \phi D + a \tag{2.2}$$

$$B_n^{MR} = (-4)^n (L^{MR})_+^{n+1/2} \tag{2.3}$$

where the superderivative $D \equiv \partial_\theta + \theta\partial$ satisfies $D^2 = \partial$, θ is the Grassmann variable ($\theta^2 = 0$) which, together with the even variable $x \equiv t_1$, define the (1|1) superspace [9] with coordinate (x, θ). The formal inverse of D is introduced by $D^{-1} = \theta + \partial_\theta \partial^{-1}$, which satisfies $DD^{-1} = D^{-1}D = 1$. The coefficients $\phi = \phi(x, \theta)$ and $a = a(x, \theta)$ are odd and even superfields on (1|1) superspace, respectively. We denote the action of the superderivative D on the superfield f by (Df).

The bi-Hamiltonian structure of the MR hierarchy has been obtained in [10] as

$$\Theta_1^{MR} : \begin{pmatrix} \delta a \\ \delta\phi \end{pmatrix} \rightarrow \begin{pmatrix} \dot{a} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} -D\partial + \phi & -\partial \\ -\partial & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta\phi \end{pmatrix} \tag{2.4}$$

$$\Theta_2^{MR} : \begin{pmatrix} \delta a \\ \delta\phi \end{pmatrix} \rightarrow \begin{pmatrix} \dot{a} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} P_{aa} & P_{a\phi} \\ P_{\phi a} & P_{\phi\phi} \end{pmatrix} \begin{pmatrix} \delta a \\ \delta\phi \end{pmatrix} \tag{2.5}$$

where the operators P_{ij} are given by

$$\begin{aligned} P_{aa} = {} & D\partial^3 - 3\phi\partial^2 + 4aD\partial + (2(Da) - 3\phi_x)\partial + 2a_x D + 3\phi(D\phi) \\ & + (Da)_x - 4a\phi - \phi_{xx} + \phi D^{-1}(Da) - (Da)D^{-1}\phi \\ & - \phi D^{-1}\phi D^{-1}\phi - \phi D^{-1}\phi_x + \phi_x D^{-1}\phi \end{aligned} \tag{2.6}$$

$$P_{a\phi} = \partial^3 - 2\phi D\partial + 4a\partial - \phi_x D + 2a_x + \phi D^{-1}(D\phi) \tag{2.7}$$

$$P_{\phi a} = \partial^3 + 2\phi D\partial + (4a - 2(D\phi))\partial + \phi_x + 2a_x - (D\phi)_x + (D\phi)D^{-1}\phi \tag{2.8}$$

$$P_{\phi\phi} = 4\phi\partial + 2\phi_x. \tag{2.9}$$

Here, following the notation of [1], the phase space for the MR theory is a pair $m = (a, \phi)$. A tangent vector at a point m is denoted by $\dot{m} = (\dot{a}, \dot{\phi})$ and a cotangent vector as a pair $\delta m = (\delta a, \delta\phi)$ where $\dot{a}$ and $\delta\phi$ are even superfields, whereas $\dot{\phi}$ and δa are odd. The inner product is defined by $\langle \delta m, \dot{m} \rangle \equiv \int \mathrm{d}x\, \mathrm{d}\theta\, (\delta a \dot{a} + \delta\phi\dot{\phi})$.

For the LM hierarchy, the Lax equation is given by

$$\partial_n L^{\mathrm{LM}} = [B_n^{\mathrm{LM}}, L^{\mathrm{LM}}] \tag{2.10}$$

with

$$L^{\mathrm{LM}} = \partial^2 - 2u\partial - ((Du) + \tau)D \tag{2.11}$$

$$B_n^{\mathrm{LM}} = (-4)^n (L^{\mathrm{LM}})_{>0}^{n+\frac{1}{2}} \tag{2.12}$$

where $\mu = \mu(x, \theta)$ and $\tau = \tau(x, \theta)$ are even and odd superfields, respectively. It should be mentioned that the LM theory discussed here is obtained from the $N = 2, \alpha = -2$ LM sKdV theory [5]. The bi-Hamiltonian structure of the LM hierarchy is also taken from [10], in component form, as [1]

$$(\Theta_1^{\mathrm{LM}})^{-1} : \begin{pmatrix} \dot{u} \\ \dot{\tau} \end{pmatrix} \to \begin{pmatrix} \delta u \\ \delta\tau \end{pmatrix} = \begin{pmatrix} D - D^{-1}\tau D^{-1} & u\partial^{-1} + D^{-1}uD^{-1} \\ \partial^{-1}u + D^{-1}uD^{-1} & D^{-1} - \partial^{-1}\tau\partial^{-1} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{\tau} \end{pmatrix} \tag{2.13}$$

$$\Theta_2^{\mathrm{LM}} : \begin{pmatrix} \delta u \\ \delta\tau \end{pmatrix} \to \begin{pmatrix} \dot{u} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} -D\partial + \tau & 2u\partial - (Du)D + 2u_x \\ 2u\partial - (Du)D + u_x & -D\partial^2 + 3\tau\partial + (D\tau)D + 2\tau_x \end{pmatrix} \begin{pmatrix} \delta u \\ \delta\tau \end{pmatrix} \tag{2.14}$$

where, similarly, the phase space of the LM theory can be represented as a set of pairs $n = (u, \tau)$. Then the tangent and cotangent vectors at a point n are represented as $\dot{n} = (\dot{u}, \dot{\tau})$ and $\delta n = (\delta u, \delta\tau)$, respectively, where $\dot{u}$ and $\delta\tau$ are even, while δu and $\dot{\tau}$ are odd. The inner product is defined by $\langle \delta n, \dot{n}\rangle \equiv \int \mathrm{d}x\, \mathrm{d}\theta\, (\delta u\dot{u} + \delta\tau\dot{\tau})$. More features about the bi-Hamiltonian structures of these two hierarchies have been tabulated in [1].

3. Gauge transformations

In [7], Inami and Kanno showed that the MR sKdV hierarchy can be related to the LM sKdV hierarchy via the following gauge transformation:

$$L_1^{\mathrm{MR}} = S_1^{-1} L^{\mathrm{LM}} S_1 \qquad S_1 = \exp\left(\int^x u\right) \tag{3.1}$$

which leads to

$$\phi_1 = (Du) + \tau \qquad a_1 = u_x - u^2 - ((Du) + \tau)(D^{-1}u). \tag{3.2}$$

They also showed that the Lax equation (2.1) of the LM theory is mapped to the Lax equation (2.10) of the MR theory under such a transformation. Hence equations (3.2) provide a gauge equivalence between these two hierarchies and now is referred to as the Inami–Kanno transformation. It can be shown that S_1^{-1} is an eigenfunction of the MR sKdV hierarchy, i.e. $\partial_n S_1^{-1} = (B_n^{\mathrm{MR}} S_1^{-1})$. Furthermore, Morosi and Pizzocchero [1] showed that the IK transformation is a canonical map, in the sense that the bi-Hamiltonian structure of the LM sKdV hierarchy is mapped to the bi-Hamiltonian structure of the MR sKdV hierarchy. That is

$$S_1'^{\dagger} (\Theta_1^{\mathrm{MR}})^{-1} S_1' = (\Theta_1^{\mathrm{LM}})^{-1} \tag{3.3}$$

$$S_1' (\Theta_2^{\mathrm{LM}}) S_1'^{\dagger} = (\Theta_2^{\mathrm{MR}}) \tag{3.4}$$

where S_1' and $S_1'^{\dagger}$ are respectively the linearized map and the transport map of the IK transformation, which satisfy

$$\langle S_1'^{\dagger}\delta m, \dot{n}\rangle = \langle \delta m, S_1'\dot{n}\rangle. \tag{3.5}$$

In fact, we can construct another transformation between the MR and LM sKdV hierarchies as follows:

$$L_2^{\mathrm{MR}} = S_2^{-1} L^{\mathrm{LM}} S_2 \qquad S_2 = D^{-1} S_1. \tag{3.6}$$

Then a simple calculation leads to

$$\phi_2 = (Du) - \tau \qquad a_2 = -u^2 - (D\tau) - ((Du) - \tau)(D^{-1}u). \tag{3.7}$$

It can be shown that the Lax equations are preserved under such transformation. Hence the transformation (3.6) also provides a gauge equivalence of the MR and LM sKdV hierarchies.

Similarly, we can show that, in this case, $\partial_n S_1 = -((B_n^{\rm MR})^* S_1)$. That means S_1 is an adjoint eigenfunction of the MR sKdV hierarchy.

Next, let us discuss the canonical property of the transformation (3.6). From equations (3.7), the linearized map S_2' and its adjoint map $S_2'^{\dagger}$ can be derived as follows:

$$S_2' = \begin{pmatrix} -2u + (D^{-1}u)D - \phi_2 D^{-1} & -D - (D^{-1}u) \\ D & -1 \end{pmatrix} \tag{3.8}$$

$$S_2'^{\dagger} = \begin{pmatrix} -2u + D^{-1}\phi_2 - D(D^{-1}u) & -D \\ -D + (D^{-1}u) & -1 \end{pmatrix}. \tag{3.9}$$

After a straightforward but tedious calculation, the Poisson structures transform as

$$S_2'^{\dagger}(\Theta_1^{\rm MR})^{-1} S_2' = -(\Theta_1^{\rm LM})^{-1} \tag{3.10}$$

$$S_2'(\Theta_2^{\rm LM}) S_2'^{\dagger} = -(\Theta_2^{\rm MR}) \tag{3.11}$$

which, comparing with (3.3) and (3.4), acquire a minus sign. It seems that this result contradicts the property of preserving the Lax equations. However, it is not the case. Since the parity of the gauge operator S_2 is odd, the Hamiltonian of the LM hierarchy $H_n^{\rm LM} = {\rm str}((L^{\rm LM})^{n+1/2})$ (up to a multiplicative constant) then is transformed to $H_n^{\rm MR} = -H_n^{\rm LM}$ due to the following property:

$${\rm str}(PQ) = (-1)^{|P||Q|}{\rm str}(QP) \tag{3.12}$$

where P and Q are any super-pseudo-differential operators with gradings $|P|$ and $|Q|$, respectively. Therefore, the gauge equivalence is compatible with the canonical property under the transformation triggered by the gauge operator S_2.

Based on the above discussions, the canonical property of the gauge transformations between the MR and LM sKdV hierarchies can be summarized as follows:

$$S_i'^{\dagger}(\Theta_1^{\rm MR})^{-1} S_i' = (-)^{|S_i|}(\Theta_1^{\rm LM})^{-1} \tag{3.13}$$

$$S_i'(\Theta_2^{\rm LM}) S_i'^{\dagger} = (-1)^{|S_i|}(\Theta_2^{\rm MR}) \qquad i = 1, 2 \tag{3.14}$$

which seems to be the supersymmetric generalization of the bosonic case [11].

4. The Bäcklund transformation and the Kupershmidt–Wilson theorem

From the IK transformation, we know that if we have a solution $\{u, \tau\}$ of the LM sKdV hierarchy, then equations (3.2) gives a solution $\{\phi_1, a_1\}$ of the MR sKdV hierarchy. Sometimes, such a transformation of one hierarchy to another is called a Miura transformation. On the other hand, equation (3.7) also gives another solution $\{\phi_2, a_2\}$ of the MR sKdV hierarchy. Hence a Darboux–Bäcklund transformation (DBT) of the MR sKdV hierarchy to itself can be constructed from these two gauge transformations. In other words, let $\{\phi_1, a_1\}$ be a solution of the MR sKdV hierarchy, then solving $\{u, \tau\}$ from (3.2) and substituting it in (3.7) we get

$$\phi_2 = -\phi_1 - 2(D^3 \ln S_1^{-1}) \tag{4.1}$$

$$a_2 = a_1 - (D\phi_1) + 2(D \ln S_1^{-1})(\phi_1 + (D^3 \ln S_1^{-1})) \tag{4.2}$$

which is simply the DBT derived in [12]. The action of the gauge operators S_1 and S_2 for the MR and LM sKdV hierarchies are shown as follows:

$$\begin{array}{ccccc} & & {\rm LM} & & \\ & {\scriptstyle S_1}\swarrow & & \searrow{\scriptstyle S_2} & \\ {\rm MR}^1 & \longleftarrow & {\rm DBT} & \longrightarrow & {\rm MR}^2. \end{array} \tag{4.3}$$

In what follows, we wish to discuss the relationship in (4.3) from the KW theorem viewpoint, in which the gauge operator S_2 plays an important and unambiguous role. From (2.11), the Lax operator L^{LM} can be factorized as follows [7]:

$$\begin{aligned} L^{\mathrm{LM}} &= \partial^2 - 2u\partial - ((Du) + \tau)D \\ &= (D - \Phi_1)(D - \Phi_1 - \Phi_2)(D - \Phi_2)D \end{aligned} \tag{4.4}$$

where u and τ can be expressed in terms of the superfields Φ_i as

$$u = \tfrac{1}{2}[(D\Phi_1) + (D\Phi_2) - \Phi_1\Phi_2] \tag{4.5}$$

$$\tau = \tfrac{1}{2}[\Phi_{2x} - \Phi_{1x} - (D\Phi_1)\Phi_2 - \Phi_1(D\Phi_2)]. \tag{4.6}$$

We notice that the second Hamiltonian structure of the LM theory can be simplified under the factorization (4.4). From equations (4.5) and (4.6), it can be straightforwardly shown that

$$\Theta_2^{\mathrm{LM}} = \left[\frac{\partial(u,\tau)}{\partial(\Phi_1,\Phi_2)}\right]\begin{pmatrix} 0 & 2D \\ 2D & 0 \end{pmatrix}\left[\frac{\partial(u,\tau)}{\partial(\Phi_1,\Phi_2)}\right]^{\dagger} \tag{4.7}$$

where the Fréchet derivative can be calculated as

$$\left[\frac{\partial(u,\tau)}{\partial(\Phi_1,\Phi_2)}\right] = \begin{pmatrix} -\frac{1}{2}(D + \Phi_2) & -\frac{1}{2}(D - \Phi_1) \\ -\frac{1}{2}(\partial + \Phi_2 D + (D\Phi_2)) & \frac{1}{2}(\partial - \Phi_1 D - (D\Phi_1)) \end{pmatrix} \tag{4.8}$$

and $\left[\partial(u,\tau)/\partial(\Phi_1,\Phi_2)\right]^{\dagger}$ is its formal adjoint.

Now applying the IK transformation to (4.4), we obtain the multiplicative form of the Lax operator L_1^{MR} as

$$L_1^{\mathrm{MR}} = (D - \Psi_1)(D - \Psi_2)(D - \Psi_3)(D - \Psi_4) \tag{4.9}$$

where the superfields Ψ_i are given by

$$\Psi_1 = \tfrac{1}{2}((D^{-1}\Phi_1\Phi_2) + \Phi_1 - \Phi_2) \tag{4.10}$$

$$\Psi_2 = \tfrac{1}{2}((D^{-1}\Phi_1\Phi_2) + \Phi_1 + \Phi_2) \tag{4.11}$$

$$\Psi_3 = \tfrac{1}{2}((D^{-1}\Phi_1\Phi_2) + \Phi_2 - \Phi_1) \tag{4.12}$$

$$\Psi_4 = \tfrac{1}{2}((D^{-1}\Phi_1\Phi_2) - \Phi_1 - \Phi_2). \tag{4.13}$$

Note that only two of them are independent variables. The Lax equation for L_1^{MR} then can be expressed in terms of the hierarchy equations of Ψ_i.

On the other hand, if we apply the gauge transformation (3.6) to (4.4), the Lax operator L_2^{MR} is then factorized as

$$L_2^{\mathrm{MR}} = (D - \Psi_4)(D - \Psi_1)(D - \Psi_2)(D - \Psi_3) \tag{4.14}$$

which differs from L_1^{MR} only by a cyclic permutation: $\Psi_1 \mapsto \Psi_2, \ldots, \Psi_4 \mapsto \Psi_1$. Such cyclic permutation does not change the hierarchy equations of Ψ_i [13] and hence generates the DBT for the MR sKdV hierarchy itself.

5. Concluding remarks

We have shown that, in addition to the IK transformation, there is another gauge transformation between the MR and LM sKdV hierarchies. We have investigated the canonical property of this new gauge transformation and have shown that it depends on the grading (or parity) of the gauge operator. Using this new gauge transformation and the IK transformation we re-derived the DBT for the MR sKdV hierarchy. We have also given an interpretation of this new gauge transformation from the KW theorem viewpoint.

Finally, we would like to remark that the geometrical interpretation of the IK transformation discussed in [1] can also be applied to the new gauge transformation. The only thing we have to do is to choose a different cross section $\hat{\Sigma}$, which is matrix in the fibre over m of the form

$$\hat{\Sigma}(m) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ (Du) - \tau & 0 & 0 & 1 \\ -2u & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.1}$$

Then the transformation (3.7) comes out naturally from a general equation derived in [1] which describes the quotient space in the bi-Hamiltonian reduction of a loop superalgebra. Since the IK transformation was also derived from the same equation, thus equations (3.2) and (3.7) can be treated on an equal footing in the bi-Hamiltonian framework. We hope such an approach can be generalized to other super integrable systems. Work in this direction is still in progress.

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